Minimal symmetric Darlington synthesis: the real case

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Abstract. We consider the symmetric Darlington synthesis of a $p \times p$ rational symmetric Schur function S with the constraint that the extension is of size $2p \times 2p$ and we investigate what happens when we impose that S and its extension have real coefficients. In this case, under the assumption that S is strictly contractive in at least one point of the imaginary axis, we determine the an upper bound for the McMillan degree of the extension. A constructive characterization of all such extensions is provided in terms of a symmetric realization of S and of the outer spectral factor of $I_p - SS^*$.

Keywords. *symmetric Darlington synthesis, inner extension, MacMillan degree, symmetric Potapov factorization.*

I. INTRODUCTION

The Darlington synthesis problem has a long history which goes back to the time when computers were not available and the synthesis of non-lossless circuits was a hard problem: the brilliance of the Darligton synthesis was that it reduced any synthesis problem to a lossless one. In mathematical terms, given a $(p \times p)$ Schur function S, say, in the right half-plane, the problem is to imbed S into a $(m+p) \times (m+p)$-inner function $\mathcal{S}$ so that:

$$\mathcal{S} = \left(\begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S \end{array} \right), \qquad \mathcal{S}(i\omega)\mathcal{S}^*(i\omega) = I_{m+p}, \quad \omega \in \mathbb{R}. \tag{1}$$

This problem was first studied by Darlington in the case of a scalar rational S [D1], was generalized to the matrix case [B1], and finally carried over to non-rational S [A2], [D2]. An imbedding of the form (1) will be called a *Darlington synthesis* or *inner completion*, or even sometimes a *lossless extension* of S.

In particular $\mathcal{S}$ can be chosen rational, and also to have real coefficients if S does.

When S is the scattering matrix of an electric p-pole without gyrators [B1], the reciprocity law entails that S is symmetric and the question arises whether the extension $\mathcal{S}$ can also be made symmetric; this would result in a Darlington synthesis which is itself free from gyrators. In [AV] it is shown that a symmetric Darlington synthesis of a symmetric rational S indeed exists and, although one can no longer preserve the degree while keeping $m = p$, he can at least ensure that $\deg \mathcal{S} \leq 2 \deg \mathrm{S}$. The existence of a symmetric Darlington synthesis for non-rational functions has been studied in [A3], in the slightly different but equivalent setting of J-inner extensions.

In [AV] it is also shown that, by increasing the *size* m to $p+n$, where n is the degree of S, it is possible to construct a *symmetric* extension of exact degree n. However, such an increase of m is not always appropriate. In fact, although the original motivations from circuit synthesis that brought the problem of lossless imbedding to the fore are mostly forgotten today, the authors of the present paper were led to raise the above issue in connection with the modeling of Surface Acoustic Waves filters [BEGO]. In this context, physical constraints impose $m = p$, so that each block of the electro-acoustic scattering matrix $\mathcal{S}$ in (1) has to be of size $p \times p$.

It is thus natural to ask the following : *given a symmetric rational S, what is the minimal degree of a symmetric lossless extension $\mathcal{S}$?* This is the problem that we have considered in [BEGO]. Here we investigate this problem using frequency domain tools with a particular focus on the case when S and its extension are constrained to have real coefficients. We restrict our attention to the case where S is strictly contractive in at least one point of the imaginary axis. This implies that the extension will have size $2p$. For the general case, that is, with extensions of lower size, the analysis seems to be more difficult and it will possibly be treated in a subsequent paper.

In Section II we introduce some notations. In Section III we construct an inner extension preserving the degree and we characterize all inner extensions in terms of minimal ones. In Section IV we finally produce a symmetric inner $2p \times 2p$ extension of minimal degree as well as a degree n extension of dimension $2(p+1) \times 2(p+1)$. In Section V we discuss the symmetric *and conjugate symmetric* unitary extension of a rational symmetric Schur function which is conjugate symmetric (*i.e.* that has real coefficients).

II. PRELIMINARIES AND NOTATIONS

As usual, we denote by Π^+ and Π^- the right and left open half-planes and by $i\mathbb{R}$ the imaginary axis; similarly, $\mathbb{D}$ denotes the unit disk, $\mathbb{T}$ its boundary and $\mathbb{E} = \mathbb{C} \setminus \mathbb{D} \cup \{\infty\}$. Nevertheless, one important feature of the frequency domain approach is that it allows to accommodate both discrete and continuous time settings. Throughout, if M is a complex matrix, $\mathrm{Tr}\ (M)$ stands for its trace, M^T for its transpose and M^* for the transposed conjugate;

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In order to avoid the above repeated definitions in continuous and discrete time, we shall denote, following [D4], by Δ^+ the region of analyticity of our functions and by ∂ its boundary. More precisely, we will use the following table:

	Continuous time	Discrete time
Δ^+	Π^+	$\mathbb{E}$
Δ^-	Π^-	$\mathbb{D}$
∂	$i\mathbb{R}$	$\mathbb{T}$
s'	$-\bar{s}$	$1/\bar{s}$
$F^*(s)$	$F(-\bar{s})^*$	$F\left(1/\bar{s}\right)^*$
$b_\omega(s)$	$(s-\omega)/(s+\bar{\omega})$	$(s-\omega)/(1-s\bar{\omega})$

(2)

Note that $*$ has two different meanings depending on its position with respect to the variable; this slight ambiguity is common in the literature and allows for a simpler notation.

Usually, System Theory is concerned with functions having the *conjugate symmetry*: $W(\bar{s}) = \overline{W(s)}$, but we shall not make this restriction unless otherwise stated. A rational function has the conjugate-symmetry if, and only if, it has real coefficients.

We say that a $p \times p$ matrix valued function S is a *Schur function* if it is contractive in Δ^+ :

$$S(s)S(s)^* \leq I, \quad s \in \Delta^+. \tag{3}$$

Note that a rational matrix which is contractive in Δ^+ is automatically analytic there. In System Theory, a rational function whose poles lie in Δ^- is called *stable*, a rational function which is finite (resp. vanishing) at infinity is called *proper* (resp. *strictly proper*) and a Schur function also called *bounded real*.

A Schur function S is said to be *lossless* or *inner* if, in addition,

$$S(s)S(s)^* = I, \qquad s \in \partial. \tag{4}$$

Notice that if $\omega \in \Delta^+$, the Blaschke factor $b_\omega(s)$ in (2) is an inner function of McMillan degree one. Note that in continuous-time, ∞ belongs to ∂ while in discrete-time ∞ belong to Δ^+ and the corresponding inner Blaschke factor is $b_\infty = 1/s$. Every scalar rational inner function can be obtained as a product of Blaschke factors $\epsilon\, b_{\omega_1}(s)b_{\omega_2}(s)\ldots b_{\omega_n}(s)$, $\epsilon \in \mathbb{T}$, $w_i \in \Delta^+$.

Let $q(s) = (s-\omega_1)(s-\omega_2)\ldots(s-\omega_n)$, $\omega_i \in \Delta^-$ be a stable polynomial. We denote by $\widetilde{q}$ the *reflection* of q that is the polynômial such that

$$\frac{q}{\widetilde{q}} = b_{\omega_1}(s)b_{\omega_2}(s)\ldots b_{\omega_n}(s). \tag{5}$$

If $Q(s)$ is a rational $p \times p$ inner function, its determinant is a scalar inner function. Thus

$$\det Q(s) = \epsilon \frac{\widetilde{q}}{q},$$

where q is a stable polynomial (whose roots are in Δ^-). The McMillan degree of $Q(s)$ is precisely the degree of $q(s)$. Since no cancellation can occur between q and $\widetilde{q}$, it follows in particular that

$$\deg(Q_1 Q_2) = \deg Q_1 \deg Q_2,$$

whenever Q_1 and Q_2 are inner of the same size.

Let S be a $p \times p$ symmetric Schur function that is strictly contractive at infinity. We shall presently study the $(p+m) \times (p+m)$ lossless extensions $\mathcal{S}$ which are symmetric as well :

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}, \text{ with } \left\{ \begin{array}{ccc} S_{11} & = & S_{11}^T \\ S_{21} & = & S_{12}^T. \end{array} \right. \tag{6}$$

The symmetric Darlington synthesis problem has been little studied and the existing solutions require an important increase of the size of the extension [AV]. In [BEGO], it was proved that a $p \times p$ symmetric Schur matrix $S(s)$ do possess a $2p \times 2p$ symmetric lossless extension if and only if its zeros have even multiplicity. If this condition is not satisfied, given a $p \times p$ symmetric Schur matrix S of McMillan degree n, two dual extension problems can be formulated:

- either we fix the size of the extension to $2p \times 2p$ and we look for a minimal degree extension,
- or we fix the degree n of the extension and we look for a minimum size extension.

The solution obtained in [BEGO] for the first problem improves significantly the results in the literature. In this paper, we give a proof of this result using a frequency domain approach. We also show that, by increasing the *size* m to $p+1$, it is possible to construct a *symmetric* extension of exact degree n. However, these results require that we allow the extension to have complex coefficients. In section V, we specialize these results to the real case.

III. INNER COMPLETIONS

The left lower block S_{21} of the completion (6) is a spectral factor of $I - SS^*$,

$$I - SS^* = S_{21}S_{21}^*. \tag{7}$$

There are many ways to achieve such a factorization. The size $p \times m$ of a spectral factor is such that $m \geq m_0$, the normal rank of $I - SS^*$.

If $m_0 < p$, then by [D4,] we may write $S(s)$ in the form

$$S(s) = V \begin{bmatrix} I & 0 \\ 0 & S_0(s) \end{bmatrix} V^T,$$

where V is a constant unitary matrices and $S_0(s)$ a strictly contractive symmetric Schur matrix. The completion problem for $S(s)$ is thus a completion problem for the Schur matrix $S_0(s)$.

In the sequel we shall assume that the normal rank of $S(s)$ is p. Since S is rational, this amounts to say that $S(s)$ is strictly contractive at least at some point of ∂.

Remark. We can assume, without loss of generality, that $(I+S)$ is invertible in the closed right half-plane. In fact, since S is Schur, $(I+S)$ is invertible in Δ^+. Since S is rational and strictly contractive, there is a finite set of values $\gamma_1, ..., \gamma_r \in \mathbb{T}$ such that $S(i\omega_i)v_i = \gamma_i v_i$, for some $v_i \in \mathbb{C}^p$ and $\omega_i \in \delta$, $i = 1, ..., r$; then we can pick any complex number δ such that $|\delta| = 1$, $\delta \neq \gamma_1, ..., \gamma_r$ and define $S_1 := \delta S$; we can then consider the extension problem for S_1. A solution to the original problem will be obtained by the inverse transformation. Notice that $(I + S)$ is then a unit in H^∞.

In the case a function is wide inner, its completion can be easily computed using state space formulas (see e.g. [HF]); nevertheless, a frequency domain expression which makes use of the information about the entries seem to be lacking. The following lemma provides one.

Theorem 1: Let S be a $p \times p$ Schur function such that $I + S$ is invertible in H^∞. Let S_{21} be a spectral factor of $I_p - SS^*$. Every inner completion $\mathcal{S}$ of $[S_{21}\ S]$ can be written as:

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \tag{8}$$

with

$$\begin{aligned} S_{12} &= -MS_{21}^*(I + S^*)^{-1}(I + S) && (9) \\ S_{11} &= M - MS_{21}^*(I + S^*)^{-1}S_{21} && (10) \end{aligned}$$

where M is a left DSS inner factor of $(I + S)^{-1}S_{21}$, i.e.

$$MS_{21}^*(I + S^*)^{-1} \tag{11}$$

is stable. The extension $\mathcal{S}$ has same degree as $[S_{21}\ S]$ if and only if M has minimal degree.

Proof: It is easily verified that (8) provides an inner extension of $[S_{21}\ S]$. Conversely, let us prove that every inner extension can be written in this form. Let

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

be an inner completion of $[S_{21}\ S]$ and put

$$M = S_{11} - S_{12}(I_p + S)^{-1}S_{21}. \tag{12}$$

Using the fact that $\mathcal{S}$ is inner, i.e.

$$\begin{aligned} S_{11}S_{11}^* + S_{12}S_{12}^* &= I_p \\ S_{11}S_{21}^* + S_{12}S^* &= 0 \\ S_{21}S_{21}^* + SS^* &= I_p \end{aligned} \tag{13}$$

and (12), it is easily shown that:

$$S_{12} = -MS_{21}^*(I + S^*)^{-1}(I + S). \tag{14}$$

By construction, M is analytic and from (12) and (13), it is easily verified that M is inner. Now (14) can be rewritten

$$MS_{21}^*(I + S^*)^{-1} = -S_{12}(I + S)^{-1}.$$

Notice that $-S_{12}(I+S)^{-1}$ is stable whereas $S_{21}^*(I+S^*)^{-1}$ is antistable, so that M is a leftt inner factor of $S_{12}(I+S)^{-1}$. Let n' be the degree of $[S_{21}\ S]$. We claim that $\mathcal{S}$ has degree n' if and only if M and $\alpha = S_{12}(I+S)^{-1}$ are left coprime. Suppose first that R is a right inner commun factor of degree d: $M = RM_1$ and $\alpha = R\alpha_1$, α_1 in H^2.

$$\begin{aligned} \begin{bmatrix} R^* & 0 \\ 0 & I \end{bmatrix}\mathcal{S} &= \begin{bmatrix} R^* & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} M - \alpha S_{21} & -\alpha(I+S) \\ S_{21} & S \end{bmatrix} \\ &= \begin{bmatrix} M_1 - \alpha_1 S_{21} & -\alpha_1(I+S) \\ S_{21} & S \end{bmatrix} \end{aligned}$$

is an extension of $S(s)$ which has degree $n' - d$, which is impossible.

Conversely, let $\mathcal{S}(s)$ be an extension of $[S_{21}\ S]$. We know that there exists an extension $\tilde{\mathcal{S}} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ S_{21} & S \end{bmatrix}$ of $[S_{21}\ S]$ of degree n' with $\tilde{S}_{11}$ and $\tilde{S}_{12}$ left coprime. It is immediate to see that

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ S_{21} & S \end{bmatrix}$$

for some unitary function $R(s)$. Since $\tilde{S}_{11}$ and $\tilde{S}_{12}$ are left coprime there exists analytic functions X and Y such that

$$\tilde{S}_{11}X + \tilde{S}_{12}Y = I,$$

then $R = S_{11}X + S_{12}Y$ is analytic in Δ^+. Therefore R is inner and

$$\deg \mathcal{S} = \deg R + n'.$$

If $\deg \mathcal{S} = n'$, then R is a constant unitary matrix while if $\deg \mathcal{S} > n'$, then R it is a common left factor of S_{11} and S_{12}. This implies that M and α are not coprime. □

Lemma 1: Let S be a $p \times p$ Schur function of degree n strictly contractive at some point of ∂. Let

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

be an inner completion of S; Then,

$$\det \mathcal{S} = \det S \det S_{11}^{-*} \tag{15}$$

Moreover, if S_{21} is minimal size $p \times p$, then

$$\det \mathcal{S} = -\det S_{12} \det S_{21}^{-*} \tag{16}$$

Proof: First remark that S being strictly contractive at some point of ∂, neither S_{12} nor S_{21} could vanish identically and thus they are both invertible. Observe now that, using the Schur complement of S_{12} and the fact that $\mathcal{S}$ is inner (so that $S_{11} = -S_{12}S^*S_{21}^{-*}$) we have:

$$\begin{aligned} \det \mathcal{S} &= -\det S_{12} \det(S_{21} - SS_{12}^{-1}S_{11}) \\ &= -\det S_{12} \det(S_{21} + SS_{12}^{-1}S_{12}S^*S_{21}^{-*}) \\ &= -\det S_{12} \det(S_{21} + (I - S_{21}S_{21}^*)S_{21}^{-*}) \\ &= -\det S_{12} \det S_{21}^{-*} \end{aligned}$$

Samely,using the Schur complement of S and the relation $S_{21} = -SS_{12} * S_{11}^{-*}$ we have

$$\begin{aligned} \det \mathcal{S} &= \det S \det(S_{11} - S_{12}S^{-1}S_{21}) \\ &= -\det S \det(S_{11} + S_{12}S^{-1}SS_{12} * S_{11}^{-*}) \\ &= -\det S \det(S_{11} + (I - S_{11}S_{11}^*)S_{11}^{-*}) \\ &= -\det S \det S_{11}^{-*} \end{aligned}$$

Proposition 1: All rational inner extensions of a contractive rational function S, can be written on the form

$$\begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}\mathcal{S}\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where L, R and $\mathcal{S}$ are inner, and $\mathcal{S}$ is a minimal degree inner extension of S.

IV. MINIMAL DEGREE VS MINIMAL SIZE SYMMETRIC INNER COMPLETIONS

Proposition 2: [BEGO, Prop.4] Let

$$\check{S} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \tag{17}$$

be a minimal degree extension (Theorem 1) associated with the minimum phase spectral factor $\check{S}_{21}$ of $I - SS^*$. Let n_0 denote the number of zeros on the imaginary axis of $\check{S}_{21}(s)$. The extension

$$\check{\Sigma} = \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \begin{bmatrix} \check{Q} & 0 \\ 0 & I \end{bmatrix}, \quad \check{Q} = \check{S}_{21}^{-1}\check{S}_{12}^T, \tag{18}$$

is symmetric, lossless and has degree $2n - n_0$. If S is real, then also $\check{\Sigma}$ can be chosen real.

Proof: It is easily seen that $\check{Q}$ is unitary and analytic in the right half-plane, since $\check{S}_{21}$ is minimum phase. The assertion on the degree follows from (16). The matrix $\check{S}$ being inner, we have that $\check{S}_{11} = -\check{S}_{12}^{-*}S^*\check{S}_{21}$ and $\check{S}_{11}\check{Q} = -\check{S}_{12}^{-*}S^*\check{S}_{12}^T$ is clearly symmetric.

Since $\check{S}$ is inner its determinant is of the form

$$\det \check{S} = \epsilon\frac{\widetilde{q}}{q},$$

where $q(s) = (s-\omega_1)(s-\omega_2)\ldots(s-\omega_n)$, $w_i \in \Delta^-$. The polynomial q is the polynomial of poles of $\check{S}$ and thus the common denominator of all the minors of $\check{S}$. It has (polynomial) degree n. Therefore we have

$$\det \check{S}_{12} = \frac{p_{12}}{q}, \quad \det \check{S}_{21} = \frac{p_{21}}{q}.$$

Then,

$$\det \check{S}_{21}^{-*} = \frac{q^*}{p_{21}^*}.$$

In view of Lemma 1, we must have

$$\begin{array}{lll} p_{12} = (-1)^n p_{21}^*(s) & \text{if} & \Delta^+ = \Pi^+ \\ p_{12} = s^n p_{21}^*(s) & \text{if} & \Delta^+ = \mathbb{E} \end{array}$$

Finally,

$$\det Q = \det \check{S}_{21}^{-1} \det \check{S}_{12} = \frac{q}{p_{21}}\frac{p_{12}}{q} = \frac{p_{12}}{p_{21}}.$$

The polynomial p_{21} is the polynomial of zeros of $\check{S}_{21}$, and since $\check{S}_{21}$ is outer, these zeros are either in Δ^- or on ∂. The zeros on ∂ simplify between p_{21} and p_{12} and the degree of Q is $n - n_0$, where n_0 is the number of zeros of $\check{S}_{21}$ on ∂. □

This extension is the starting point for the construction of minimal degree and minimal size extensions. It possesses a very particular property among all the lossless extensions of S.

Lemma 2: Let $\check{S}$ be the lossless extension (17). Let κ denote the number of distinct zeros of $\det\check{Q}$ with odd multiplicity. Then, any symmetric lossless extension of S has degree greater than or equal to $n + \kappa$.

Proof: Let $\mathcal{S}(s)$ be any minimal degree extension of S, then

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} = \begin{bmatrix} L^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \check{S}_{11} & \check{S}_{12} \\ \check{S}_{21} & S \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \tag{19}$$

in which $R = \check{S}_{21}^{-1}S_{21}$ is analytic in Δ^+, since $\check{S}_{21}^{-1}$ is, and thus inner. While $L^* = S_{12}\check{S}_{12}^{-1}$ is analytic Δ^-, since $\check{S}_{12}^{-1}$ is, and thus L is inner. Note that the extension (20) is not always lossless.

Moreover, since $\mathcal{S}$ and $\check{S}$ have the same determinant, we must have

$$\det L = \det R.$$

Let Σ be the symmetric unitary extension associated with $\mathcal{S}(s)$ by

$$\Sigma = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}, \quad Q = S_{21}^{-1}S_{12}^T. \tag{20}$$

It is easily verified that we have $R^*\check{Q}(L^*)^T = Q$, and thus

$$\det Q = \det\check{Q}(\det R^*)^2.$$

Since only zeros of $\det\check{Q}$ with even multiplicity can be canceled, if κ denote the number of distinct zeros of $\det\check{Q}$ with odd multiplicity, then $\det Q(s)$ has McMillan degree greater than or equal to κ. Using Proposition 1, the result is easily deduced. □

The extension $\check{\Sigma}(s)$ is the worst that one can do, concerning the degree. In order to reduce the degree while keeping the extension symmetric, it is possible to factor out symmetrically Potapov factors ([P]) associated with double zeros. In the right half-plane setting that we consider in this chapter, the Potapov factors are of the form

$$B_{w,u}(s) = I + (b_w(s) - 1)\, uu^*. \tag{21}$$

Lemma 3 (Symmetric Potapov factorization): Let $T(s)$ be a $p \times p$ symmetric inner function of McMillan degree N. The following assertions are equivalent:

1) $T(s)$ has a zero ω of *multiplicity* strictly greater than 1
2) there exists a unit vector which satisfies the conditions

$$\begin{array}{rcl} T(\omega)u & = & 0 \\ u^T T'(\omega)u & = & 0 \end{array}$$

3) the matrix $T(s)$ can be factored as

$$T(s) = B_{\omega,u}(s)^T\, R(s)\, B_{\omega,u}(s)$$

for some rational inner matrix $R(s)$ of degree $N-2$

This result makes use of Takagi's factorization [HJ, Cor. 4.4.4] which is a special singular value decomposition for symmetric matrices: a symmetric matrix Λ can be written in the form

$$\Lambda = U^T\Delta U, \tag{22}$$

[in which Δ is a positive diagonal matrix and U is unitary.

If some partial multiplicity is greater than 1, say $\sigma_i(\omega)$, and $V(s)$ is the left unimodular matrix in the Smith-McMillan factorization of $T(s)$, then we may choose for u the ith column vector of $V(\omega)$. Otherwise, the Takagi factorization of $T'(\omega)$ provides the vector u.

Theorem 2 (Minimal degree symmetric extension): Let S be a symmetric Schur function, strictly contractive at infinity and let $\check{\Sigma}(s)$ be its minimal extension (18) with $\check{S}_{21}$ minimum phase ; define $\check{Q} := \check{S}_{21}^{-1}\check{S}_{12}^T$, and let κ be the number of distinct zeros of $\det \check{Q}$ with odd algebraic multiplicity. Then $S(s)$ has a symmetric inner completion of degree $n+\kappa$. This extension of S has minimal degree among all the symmetric extensions of S.

This result is proved by a recursive application of the symmetric Potapov factorization (Lemma 3). Since $\check{Q}(s)$ has $n-n_0=\kappa+2l$ zeros, we can perform l iterations. The zeros of $\check{Q}(s)$ lying within the p first columns of $\check{\Sigma}(s)$, the Potapov factors are all of the form $B_{w,u}\oplus I$. We finally get

$$\check{\Sigma} = \begin{bmatrix} B^T & 0 \\ 0 & I \end{bmatrix} \Sigma \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix},$$

where B, the product of the Potapov factors, has degree l and $\Sigma(s)$ is a symmetric lossless extension of S of degree $n+\kappa$.

In [AV] it was shown that it is possible to construct a symmetric extension of exact degree n by increasing the size of the extension to $2p+n$. This result can be significantly improved. Indeed, we may extend the inner matrix (18) into a $(2p+1)\times(2p+1)$ matrix of degree $n+2(n-n_0)$

$$\hat{\Sigma} := \begin{bmatrix} \tilde{S}_{11}Q & 0 & \tilde{S}_{12} \\ 0 & \det Q & 0 \\ \tilde{S}_{12}^T & 0 & S \end{bmatrix}.$$

The matrix $Q\oplus \det Q$ has exactly $n-n_0$ double zeros. But then, in view of Lemma 3, we can obtain a reduction of degree by $2(n-n_0)$.

Theorem 3 (Minimal dimension symmetric extension): Let $S(s)$ be a strictly contractive symmetric $p\times p$ Schur function of McMillan degree n. Then $S(s)$ has a symmetric inner extension of dimension $(2p+1)\times(2p+1)$ and McMillan degree n.

V. THE REAL CASE

A. Real symmetric Potapov factorization: complex zeros

Lemma 4: Let T be a symmetric inner function. Suppose B_1 and B_2 are Blaschke factors such that $\det B_1$ and $\det B_2$ have no common zeros, and that T factors as

$$\begin{aligned} T &= B_1^T T_1 B_1 \\ T &= B_2^T T_2 B_2 \end{aligned} \qquad (23)$$

with T_1 and T_2 analytic. Then $T = B^T T_0 B$ where T_0 is analytic and B is the Least Common Left Multiple (LCLM) of B_1 and B_2.
If T has real coefficients, and $B_2=\bar{B}_1$, then B and T_0 are real.

Proof: Let $\tilde{B}_2 := BB_1^*$ and $\tilde{B}_1 := BB_2^*$; since B is the LCLM of B_1, B_2, the functions $\tilde{B}_1$ and $\tilde{B}_2$ are left coprime and have no common zeros; moreover, $\det B_1 = \det \tilde{B}_1$ and $\det B_2 = \det \tilde{B}_2$. Using the equality $B_2B_1^* = \tilde{B}_1^*\tilde{B}_2$,

$$B_1^T T_1 = TB_1^* = B_2^T T_2 B_2 B_1^* = B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 \qquad (24)$$

The term on the right-hand side is analytic, and $\tilde{B}_2$ and $\tilde{B}_1$ are left coprime. This means that there exists stable matrix functions X, Y such that

$$\tilde{B}_1^*\tilde{B}_2 Y = \tilde{B}_1^* - X. \qquad (25)$$

Multiplying the last term of (24) by Y, we get:

$$B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 Y = B_2^T T_2 \tilde{B}_1^* - B_2^T T_2 X,$$

so that $B_2^T T_2 \tilde{B}_1^*$ is analytic; B_2^T is an inner function with no zeros in the poles of $\tilde{B}_1^*$, it has full rank at those points. Therefore also $T_2\tilde{B}_1^*$ is analytic.

Consider now

$$T_1 = (B_1^T)^* B_2^T T_2 \tilde{B}_1^* \tilde{B}_2 = \tilde{B}_2^T (\tilde{B}_1^*)^T T_2 \tilde{B}_1^* \tilde{B}_2$$

In view of (25) we can write:

$$\begin{aligned} & Y^T T_1 Y \\ = \ & (\tilde{B}_1^* - X)^T T_2 (\tilde{B}_1^* - X) \\ = \ & (\tilde{B}_1^*)^T T_2 \tilde{B}_1^* - X^T T_2 \tilde{B}_1^* - (\tilde{B}_1^*)^T T_2 X + X T_2 X \end{aligned}$$

which implies that $T_0 = (\tilde{B}_1^*)^T T_2 \tilde{B}_1^*$ is analytic, since all the other terms are.

If T has real coefficients and $B_2 = \bar{B}_1$, since B is the LCLM of B_1 and B_2, it is invariant under conjugation, and so it must have real coefficients, and thus so does T_0. □

The cancellation of double zeros can, in some situations, also be extended to the real case; in particular, if $\det Q$ is real and has complex conjugate double zeros, Lemma 4 can be used to reduce the degree of the extension.

B. Real symmetric Potapov factorization: real zeros

We now assume that $T(s)$ is real, symmetric and inner. We recall that the signature $\sigma(A)$ of an Hermitian matrix A is equal to the number of its positive eigenvalues minus the number of its negative eigenvalues. Let ω be a real zero of $T(s)$. Suppose that $T(s)$ has symmetric Potapov factorization

$$T(s) = B(s)^T R(s) B(s)$$

in which $B(s)$ has only a pole at ω'. By the *Sylvester's law of inertia*, for any $\xi \neq \omega$ real , $T(\xi)$ and $R(\xi)$ have the same signature: $\sigma(T(\xi)) = \sigma(R(\xi))$. Define

$$I_\omega(T) = \lim_{\epsilon\to 0^+} \frac{\sigma(T(\omega+\epsilon)) - \sigma(T(\omega-\epsilon))}{2} \qquad (26)$$

Notice that $I_\omega(T)$ is not necessarily the signature of $T(\omega)$ (this is the case only if T is non singular in ω).

Lemma 5: Let

$$T(s) = \hat{T}(s)B(s) = \begin{bmatrix} \hat{T}_{11}(s) & \hat{T}_{12}(s) \\ \hat{T}_{21}(s) & \hat{T}_{22}(s) \end{bmatrix} \begin{bmatrix} b_\omega(s)I & 0 \\ 0 & I \end{bmatrix}$$

with $\hat{T}_{11}$ invertible in ω. Then $\sigma(\hat{T}_{11}(\omega))$ is given by

$$\sigma(\hat{T}_{11}(\omega)) = I_\omega(\hat{T}B) - I_\omega(\hat{T}_{22} - \hat{T}_{21}\hat{T}_{11}^{-1}\hat{T}_{12}) \qquad (27)$$

Proof: since $T = \hat{T}B$ is symmetric, $\hat{T}_{12} = \hat{T}_{21}b_\omega$ and thus the congruence transformation $\begin{bmatrix} I & 0 \\ -T_{21}T_{11}^{-1} & I \end{bmatrix}$ yields:

$$TB \simeq \begin{bmatrix} \hat{T}_{11}b_\omega & 0 \\ 0 & \hat{T}_{22} - \hat{T}_{21}\hat{T}_{11}^{-1}\hat{T}_{12} \end{bmatrix}$$

and, at each point s, we have

$$\begin{aligned} \sigma(\hat{T}(s)B(s)) &= \sigma(\hat{T}_{11}(s)b_\omega(s)) \\ &+ \sigma(\hat{T}_{22}(s) - \hat{T}_{21}(s)\hat{T}_{11}^{-1}(s)\hat{T}_{12}(s)) \end{aligned}$$

Since $\hat{T}_{11}$ is invertible in ω and $b_\omega(s)$ has the same sign as $s-\omega$ in a neighborhood of ω,

$$\lim_{\epsilon\to 0}\sigma[(\hat{T}_{11}b_\omega)(\omega+\epsilon)] = \sigma(\hat{T}_{11}(\omega))$$
$$\lim_{\epsilon\to 0}\sigma[(\hat{T}_{11}b_\omega)(\omega-\epsilon)] = -\sigma(\hat{T}_{11}(\omega))$$

we get

$$\sigma(\hat{T}_{11}(\omega)) = I_\omega(\hat{T}_{11}b_\omega) \tag{28}$$

□

We can now use this result to obtain the information about how much we can reduce the degree in ω directly from T. We need, to this end, the following definition.

Definition 1: We say that k is the multiplicity of ω as a non reducible zero for a symmetric completion $\mathcal{S}$ of the symmetric Schur function S if, for any Blaschke factor with a zero in ω, such that $\mathcal{S}' = (B^T)^*\mathcal{S}B^*$ is analytic in Δ^+, the function $\mathcal{S}'$ has the property that its first block S'_{11} has a zero of multiplicity at least k in ω.

Lemma 6: Suppose

$$\mathcal{S} = \check{\mathcal{S}}\begin{bmatrix} Q & 0 \\ 0 & I\end{bmatrix} = \begin{bmatrix} \check{S}_{11}Q & \check{S}_{12} \\ \check{S}_{12}^T & S\end{bmatrix},$$

where $\check{\mathcal{S}}$ is the minimum-phase completion, and assume $\check{S}_{11}$ is not vanishing in ω, a real zero of Q. Then the multiplicity of ω as non reducible zero for $\mathcal{S}$ is $I_\omega(\mathcal{S})$.

Proof: if B is a Blaschke factor associated to a zero of even partial muliplicity in ω, we have

$$\sigma(\mathcal{S}(s)) = \sigma((B^T)^*\mathcal{S}(s)B^*(s))$$

and thus, without loss of generality, we can assume $Q = Q_1B_0$, where Q_1 has no zero in ω and B_0 is a Blaschke of the form $\begin{bmatrix} b_\omega(s)I & 0 \\ 0 & I\end{bmatrix}$. But then, in view of Lemma 5, and the fact that $\check{S}_{11}Q_1$ is invertible in ω (and thus it does contribute to change inertia in ω), we have, for s in a real neighbourhood of ω

$$\sigma\left(\check{S}_{11}Q_1\begin{bmatrix} b_\omega(s)I & 0 \\ 0 & I\end{bmatrix}\right) = \sigma(\check{\mathcal{S}}Q) = \sigma(\mathcal{S})$$

□

Notice that if S and S^* have neither zeros nor poles in $\omega\in\mathbb{R}$, we can conclude that the minimum-phase completion $\check{\mathcal{S}}$ is such that $\check{S}_{11}$ has no zero in ω. In fact, from (15), we have that $\det\check{S}_{11} = \det\check{S}^*\det\check{\mathcal{S}}$; if $\det\check{S}_{11}$ has a zero in ω but no pole, $\check{\mathcal{S}}$ has a zero in ω. Since $\check{\mathcal{S}}$ is an completion of S with the same degree, it has the same poles as S. Since it is inner, its zeros are the poles of S^* and we get a contradiction. On the other hand, $\check{S}_{11}$ can not have a zero pole cancellation, because its poles are those of S. Hence the conclusion.

C. Minimal symmetric real inner extension

We finally have a result about the degree of a real extension. Let $\mathcal{S}$ be a symmetric completion of S. Assume that if ω is a real zero of Q, then it is not a zero of S_{11} and define, for each ω, $I_\omega(\mathcal{S})$ as in (26). Set

$$n_r(\mathcal{S}) := \sum_\omega I_\omega(\mathcal{S})$$

Proposition 3: Let $\mathcal{S}$ be the real symmetric extension of the real symmetric Schur function S (of degree n) of the form $\check{\mathcal{S}}Q$ and let $n_c(\mathcal{S})$ be the number of complex zeros of Q with odd multiplicity. Then S has a symmetric real coefficients inner extension of dimension $(2p)\times(2p)$ and McMillan degree $n + n_c(\mathcal{S}) + n_r(\mathcal{S})$.

D. Degree n real extension of higher dimension

As in the complex case, we can try to increase the dimension beyond $2p$ while keeping the degree equal to n. Results by Vongpanitlerd (1970) (see e.g. [AV]) show that there exists such an extension of dimension $2p+n$. Nevertheless it's not difficult to see that a better bound can be achieved. In fact, we can always write an extension $\mathcal{S}$ in Smith-McMillan form as

$$\mathcal{S} = \pi^T\delta^{-1}\sigma_{sm}\pi$$

where δ and σ_{sm} are diagonal polynomial and π is a real unimodular polynomial matrix. Since $\mathcal{S}$ is inner, its zeros are antistable and thus $S_{zeros} := \sigma_{sm}\sigma_{sm}^{-*}$ is inner. Since the reduction to the Smith-McMillan form of $\mathcal{S}$ does not change its signature, the previous results show that, if we take

$$\mathcal{S}_e := \begin{bmatrix} \mathcal{S} & \\ & -S_{zeros}\end{bmatrix}$$

we easily see that $\mathcal{S}_e$ has only double zeros and the residual matrix of the real zeros has zero signature. Thus, using Lemma 4, it can be reduced to an extension of degree n.

In fact, this reduction might still not be minimal. For a minimal one we have the following:

Theorem 4: Let $\mathcal{S}_{sm}$ be the Smith-McMillan form of T and factor it as

$$\mathcal{S}_{sm} = \delta^{-1}\tau_c\tau_{rd}\tau_{rs}$$

where all the matrices are diagonal, δ is the denominator matrix, τ_c contains all the complex zeros, τ_{rd} contains the highest even number of real zeros of geometric multiplicities greater than one and τ_{rs} has only real zeros with partial multiplicities equal to 1. Let r be the number of non constant diagonal entries of τ_{rs} Then there exists an extension of dimension $2p+r$ and degree n.

Proof: Set

$$S_{mz} := \tau_{rs}\tau_{rs}^{-*}\begin{bmatrix} I_{r-1} & \\ & \det\frac{\tau_c}{\tau_c^*}\end{bmatrix}$$

The zeros of τ_c are all complex and will not change the signature. Then

$$\mathcal{S}_e := \begin{bmatrix} \mathcal{S} & \\ & -S_{mz}\end{bmatrix}$$

has the wanted dimension, double zeros with the right signature and thus it can be reduced to degree n. □

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